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Modelling and validation of a propellant mixer for controller design ☆

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Abstract

A mixing chamber used in rocket engine testing at the NASA Stennis Space Center is modelled by a system of two nonlinear ordinary differential equations. The mixer is used to condition the thermodynamic properties of cryogenic liquid propellant by controlled injection of the same substance in the gaseous phase. The three inputs of the mixer are the positions of the valves regulating the liquid and gas flows at the inlets, and the position of the exit valve regulating the flow of conditioned propellant. Mixer operation during a test requires the regulation of its internal pressure, exit mass flow, and exit temperature. A mathematical model is developed to facilitate subsequent controller designs. The model must be simple enough to lend itself to subsequent feedback controller design, yet its accuracy must be tested against real data. For this reason, the model includes function calls to thermodynamic property data. Some structural properties of the resulting model that pertain to controller design, such as uniqueness of the equilibrium point, feedback linearizability and local stability are shown to hold under conditions having direct physical interpretation.

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The existence of fixed valve positions that attain a desired operating condition is also shown. Validation of the model against real data is likewise provided.

1. Introduction

The NASA John C. Stennis Space Center (SSC) conducts extensive ground-based testing and flight certification of rocket engines, in particular, of the Space Shuttle Main Engine (SSME). Combustion chambers and turbomachinery related to rocket engines are also tested at SSC. This work is part of an on-going effort to develop a software package providing flexibility in simulation and control tasks [15,4,5,9] frequently found in test operations at SSC. Test conditions require that liquid propellants, namely liquid oxygen and liquid hydrogen (LH2) be supplied to the engine or component at very precise conditions of temperature, pressure and mass flow rate. An excess or deficiency in any of these three flow parameters may result in damaged components or in a sub-optimal test. To achieve the required conditions, the delivery system includes a mixing chamber, henceforth referred to as "mixer". The mixer subsystem is depicted in Fig. 1. LH2 is stored in the run tank, which is kept at a constant pressure by an independent control loop. Gaseous hydrogen (GH2) is stored in high pressure bottles, and has a higher temperature than the LH2. One control valve, referred to as the "liquid valve" is used to manipulate the flow of LH2 into the mixer. There is also a "gas valve" and an "exit valve". The positions of the valves constitute the only control variables for the mixer subsystem. A mixer control system must achieve tracking and regulation of mixer outputs to desired values. The outputs of interest are the mixer pressure, exit flow temperature, and exit mass flow. A dynamic model and controller are intended to replace the current method of operation of the mixer, which uses only a steady-state thermodynamic model and extensive heuristics. The use of the modelling and control techniques described herein is expected to provide great flexibility and better mixer performance. In this article, focus is placed on the mathematical modelling stage, together with an analysis of model properties and data validation.

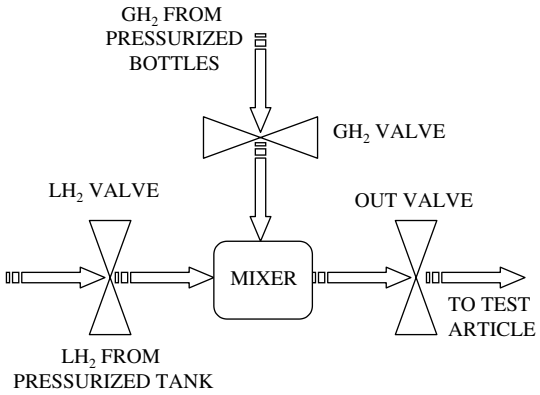

Fig. 1. The mixer subsystem.

2. Transient thermodynamics model of the mixer

2.1. Units

All expressions and numerical quantities are in SI units, with pressures in MPa, densities in kg/m^3, internal energies and enthalpies in kJ/kg, temperatures in K and mass flow rates in kg/s.

2.2. Valve models

The flow w_l of LH2 through the liquid valve depends on the pressure difference across the valve, the density of the source LH2 and the liquid valve opening coefficient C_{vl} according to [3]

$$w_l = C_1 C_{vl} \sqrt{(P_l - P)\rho_l}, \tag{1}$$

where P_l is the source LH2 pressure and P is the mixer pressure in psia; ρ_l is the source LH2 density in lbm/ft^3, and $C_1 = 2.404 \times 10^{-2}$. All valve opening coefficients are dimensionless. For gas flow, there is a separate expression [3]

$$w_g = \begin{cases} C_2 C_{vg} \sqrt{P_g^2 - P^2} \left(\frac{\sqrt{T_g}\rho_g}{P_g} \right), & \text{when } P_g < 2P, \\ C_4 C_{vg} \sqrt{T_g}\rho, & \text{when } P_g \geqslant 2P \text{ (choked flow)}, \end{cases} \tag{2}$$

where T_g is the source GH2 temperature, P_g is the source GH2 pressure, and C_{vg} is the gas valve opening coefficient. In SI units $C_2 = 1.086 \times 10^{-3}$ and $C_4 = 9.2135 \times 10^{-4}$. Specifications for the conditioned exit flow require that the propellant be in the liquid phase. Therefore it is assumed that the form of Eq. (1) is also valid for the exit flow, namely

$$w_e = C_3 C_{ve} \sqrt{(P - P_s)\rho}, \tag{3}$$

where $C_3 = C_1$, ρ and P are the mixer density and pressure, respectively, and P_s is the pressure at the outlet of the exit valve.

2.3. Mass and energy balances

The set of two nonlinear differential equations which constitute the model can be derived from the conservation of mass for compressible flow and from the First Law of Thermodynamics for a control volume including transient terms [16]. Upon manipulations, the equations may be expressed in terms of internal energy and density derivatives as

$$\dot{\rho} = \frac{1}{V}(w_l + w_g - w_e), \tag{4}$$

$$\dot{u} = \frac{1}{\rho V}\left[w_l(h_l - u) + w_g(h_g - u) - w_e(h_e - u) \right], \tag{5}$$

where V is the fixed mixer volume, h_l, h_g and h_e are the source liquid, source gas, and exit enthalpies, respectively. The mass flows w_i are functions of pressure, and are calculated according to Eqs. (1) and (2).

2.4. State variable model

Further manipulations are required to put the above physical model in state-space form. It is assumed that enthalpy is conserved across the exit valve, and the definition [16] $h = u + CP/\rho$, where C is a constant depending on the choice of units, is used. For the chosen units, $C = 1 \times 10^3$. The resulting model is compactly expressed as

$$\dot{\rho} = f_1(P(\rho,u),\rho)C_{vl} + f_2(P(\rho,u),\rho)C_{vg} + f_3(P(\rho,u),\rho)C_{ve}, \tag{6}$$

$$\dot{u} = g_1(P(\rho,u),\rho,u)C_{vl} + g_2(P(\rho,u),\rho,u)C_{vg} + g_3(P(\rho,u),\rho,u)C_{ve}, \tag{7}$$

where

$$\begin{aligned}
f_1 &= C_1'\sqrt{(P_l - P)\rho_l},\\
f_2 &= C_2', \quad \text{if } P_g \geqslant 2P,\\
f_2 &= C_4'\sqrt{P_g^2 - P^2}, \quad \text{if } P_g < 2P,\\
f_3 &= C_3'\sqrt{(P - P_s)\rho},\\
g_1 &= f_1\left(\frac{h_l - u}{\rho}\right),\\
g_2 &= f_2\left(\frac{h_g - u}{\rho}\right),\\
g_3 &= C_3''\sqrt{\frac{P - P_s}{\rho}}\frac{P}{\rho}.
\end{aligned}$$

The constants are given by $C_1' = C_1/V$, $C_2' = C_4\sqrt{T_g}\rho_g/V$, $C_3' = -C_3/V = -C_1/V$, $C_3'' = CC_3'$ and $C_4' = C_2\sqrt{T_g}\rho_g/V$. It is to be noted that, given initial conditions of internal energy and density, the numerical solution of Eqs. (6), (7) requires that the mixer pressure P be known at all times as a function of ρ and u. Such computation poses difficulties and requires special routines. Indeed, strictly speaking, density and internal energy do not completely determine pressure. Thermodynamic data for a variety of substances shows that one may find density-energy-pressure triples that have the same densities and energies with distinct corresponding pressures. Although it is possible, in principle, to find correlations that link the variables in limited ranges, the number of separate expressions and their mathematical form makes this method inadmissible in the development of a controls model. Fortunately, the errors introduced by assuming that pressure is a function of density and internal energy are fairly small in the expected range of mixer operation. The selection of a pressure based on ρ and u from thermodynamic data is not trivial, if correlations are not to be used. The available data consists of two tables, one giving density and the other energy, when pressure and temperature are known. The routines must then perform a reverse

look-up of the tables. The details of how these routines work are out of the scope of this paper, and it suffices to say that they have been proved to be accurate by using them to obtain pressure and temperature from density and energy and then recovering input data by using the original tables, with acceptably small errors. For further details, readers are referred to [7].

2.5. Output definitions

Mixer operation requires the simultaneous tracking of exit temperature, exit flow, and mixer pressure. Exit temperature and mixer pressure are functions of the energy and density states. Computation of these functions requires the intervention of thermodynamic tables and interpolation algorithms which cannot be represented in closed form. However [7], to each exit temperature and mixer pressure combination in the expected range of operation there corresponds a unique value of the state $[\rho, u]^{\mathrm{T}}$. Therefore, it is convenient to specify density and energy as outputs, along with exit mass flow. A real-time control system should perform pre-processing of commanded mixer outputs to obtain the corresponding desired state values. With these output definitions, the controls model becomes

$$\begin{cases} \begin{bmatrix} \dot{\rho} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} f_1(\rho,u) & f_2(\rho,u) & f_3(\rho,u) \\ g_1(\rho,u) & g_2(\rho,u) & g_3(\rho,u) \end{bmatrix} \begin{bmatrix} C_{\mathrm{vl}} \\ C_{\mathrm{vg}} \\ C_{\mathrm{ve}} \end{bmatrix}, \\ y = [\rho,\ u,\ -Vf_3 C_{\mathrm{ve}}]^{\mathrm{T}}, \end{cases} \tag{8}$$

where $[\,C_{\mathrm{vl}} \quad C_{\mathrm{vg}} \quad C_{\mathrm{ve}}\,]^{\mathrm{T}}$ is the control vector and y is the output vector.

3. Model properties

3.1. Uniqueness of equilibrium

The equilibrium point indicates the steady values of the density and internal energy of the mixer when the control valves are set at fixed positions, for a given set of input flow properties. For given values of the input fluid properties and C_{v} coefficients, setting $\dot{\rho}_{\mathrm{cv}} = 0$ establishes that $\dot{m}_{\mathrm{e}} = \dot{m}_{\mathrm{g}} + \dot{m}_{\mathrm{l}}$ and results in an expression relating the equilibrium density $\bar{\rho}_{\mathrm{cv}}$ to the equilibrium mixer pressure $\overline{P}$

$$\bar{\rho}_{\mathrm{cv}} = \rho(\overline{P}). \tag{9}$$

Setting $\dot{u}_{\mathrm{cv}} = 0$ gives the equilibrium exit enthalpy $h_{\mathrm{e}} = h_{\mathrm{cv}}$ in terms of the input enthalpies and the mass flows:

$$\bar{h}_{\mathrm{cv}} = \frac{\dot{m}_{\mathrm{l}} h_{\mathrm{l}} + \dot{m}_{\mathrm{g}} h_{\mathrm{g}}}{\dot{m}_{\mathrm{l}} + \dot{m}_{\mathrm{g}}} = \bar{h}(P). \tag{10}$$

The above enthalpy must match the thermodynamic property data at $\overline{P}$ and $\bar{\rho}_{\mathrm{cv}}$ of Eq. (9), that is

$$\bar{h}_{\mathrm{cv}} = h_{\mathrm{th}}(\bar{\rho}_{\mathrm{cv}}, \overline{P}).$$

Substituting Eqs. (9) and (10) into the above equation results in a single expression which gives the equilibrium pressure:

$$\bar{h}(P) = h_{\text{th}}(\rho(\overline{P}), \overline{P}). \tag{11}$$

A graphical interpretation of the equilibrium solution is shown in Fig. 2. The curve $\rho\overline{P}$ vs. $\overline{P}$ has been drawn on the base plane. This plane curve is mapped to a space curve by the thermodynamic property function h_{th}. The equilibrium exit enthalpy $\bar{h}(P)$ is a function of pressure only and the corresponding surface has also been graphed. The point where the space curve pierces the surface is the equilibrium point of the system. The monotonicity of both curve and surface indicates that there exists only one equilibrium point in the range of interest of actual mixer operation. An iterative procedure is developed in [7] that calculates the equilibrium point, given thermodynamic parameters and fixed control inputs.

3.2. Valve positions for a prescribed operating condition

Having three independent controls in a two-state model allows the selection of steady values for the states, and, in addition, an extra degree of freedom is available. This degree of freedom can be used to fix the exit mass flow with the desired thermodynamic properties, as shown next. Suppose

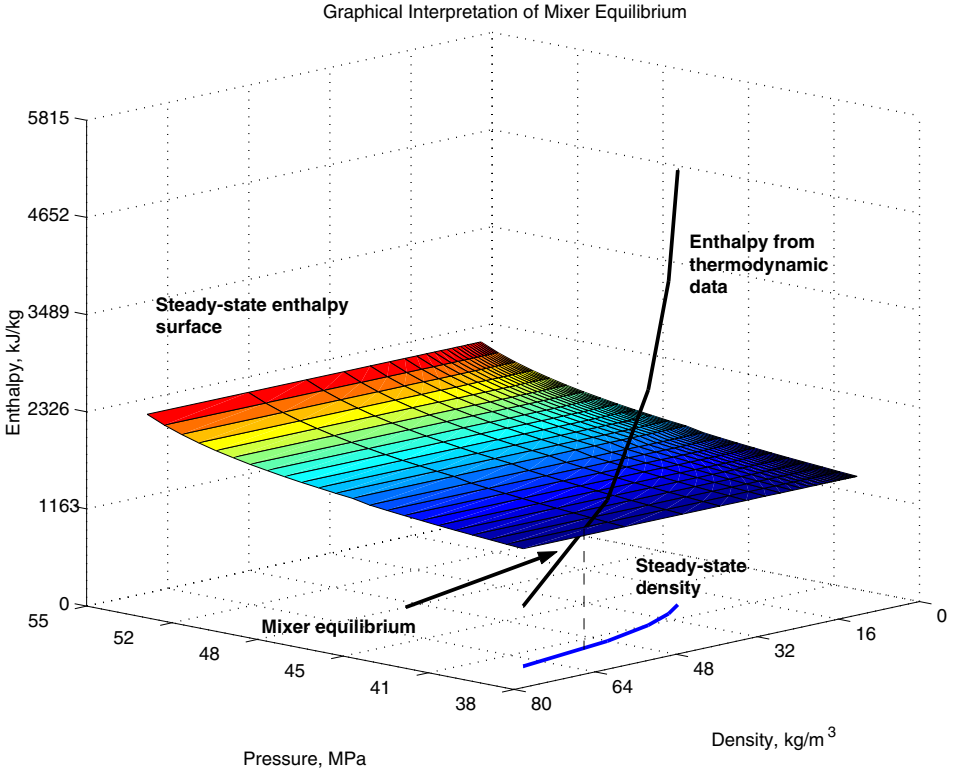

Fig. 2. Mixer equilibrium.

it is desired to have a given exit mass flow, with prescribed temperature (measured at the outlet of the exit valve). Let these quantities be denoted by w_e and T_e. The back pressure P_s at the outlet valve exit is assumed to be constant. The problem is to determine the valve coefficients that achieve this. Two degrees of freedom are used for w_e and T_e, and the third one is used to meet a desired mixer operating pressure, P. The back pressure P_s and T_e determine the enthalpy h, which is constant across the valve. Therefore, the enthalpy and mixer pressure at the exit valve inlet are known and, in turn, determine the mixer density from thermodynamic data. A static energy and material balance gives the required input flows that achieve a prescribed flow-exit enthalpy combination. The following formulas are straightforward to derive:

$$w_l = \frac{w_e(h - h_g)}{h_l - h_g}, \qquad w_g = \frac{w_e(h_l - h)}{h_l - h_g}, \tag{12}$$

where h_g and h_l are the gas and liquid supply enthalpies, respectively. Using the exit flow and the input equilibrium flows from Eq. (12), the three valve coefficients are determined from Eqs. (1) and (2). This procedure is complementary to the one described in the previous section, which calculates the equilibrium point.

4. Zero dynamics and feedback linearizability

Of the control techniques that are suitable for the present model structure, feedback linearization [6,14] is the simplest and most direct. Input/Output linearization consists in finding a coordinate transformation which results in a system which is linear between the new inputs and the outputs. Linear controller design can then be applied to the transformed system. The technique, however, may have disadvantages. Specifically, the complexity introduced by the cancellation of all nonlinear dynamics may be to high for a realistic implementation with limited computational resources. The other significant disadvantage is that all state measurements are required. Peaking of control signals is also a factor of concern, especially when actuator saturation and rate limitations are present. Linearizability analysis, however, is worth examining, since essential features of the model appear during the analysis and provide a basis for other controller design techniques. Other control techniques for the mixer and related systems are found in [1,11,2,10,8,9,7].

4.1. The concept of zero dynamics

A fair mathematical discussion of the concept of zero dynamics is out of scope and can be found in standard sources such as [6,14]. Intuitively, when the order of the linear system arising from the input transformation is less than the order of the original system, the remaining *zero dynamics* can be realized by an appropriate set of state variables. These variables represent some physical aspect of the system and must be kept bounded. Unstable zero dynamics constitute a fundamental limitation in several control design schemes, including feedback linearization. When a system has stable zero dynamics, it is said to be *minimum-phase*, a term which generalizes the concept of right-half plane zeros in linear systems.

4.2. Relative degree and input integration

The *partial relative degree* [14] of an output of a system is the order of the lowest derivative of the output which is directly affected by at least one of the inputs. The model of Eq. (8) has a characteristic which prevents direct application of input/output linearization theory. It is seen that the third component of y is algebraically related to one of the control inputs, namely C_{ve}. This implies that the partial relative degree of y_3 is zero. Conceivably, one could use a time-explicit control of the form

$$C_{\mathrm{ve}}(t,\rho,u) = -\frac{y_{3d}(t)}{Vf_3(P(\rho,u),\rho)}$$

to attain perfect tracking of the output mass flow rate and use the two remaining controls to force the flow to have the desired pressure and temperature. This approach suffers from the drawback of not employing feedback and therefore of being not self-correcting or robust. One way to get around this problem is to augment the exit valve channel with an integrator, that is, let

$$\dot{C}_{\mathrm{ve}} = v,$$

where v is a new control input. Now C_{ve} is regarded as a state, and the resulting system is of third order, with three inputs and outputs. If the arguments of functions g_i and f_i are dropped from the notation, the new system equations become

$$\dot{\rho} = f_1 C_{\mathrm{vl}} + f_2 C_{\mathrm{vg}} + f_3 C_{\mathrm{ve}}, \tag{13}$$

$$\dot{u} = g_1 C_{\mathrm{vl}} + g_2 C_{\mathrm{vg}} + g_3 C_{\mathrm{ve}}, \tag{14}$$

$$\dot{C}_{\mathrm{ve}} = v, \tag{15}$$

$$y = [\,\rho \quad u \quad -Vf_3 C_{\mathrm{ve}}\,]^{\mathrm{T}}. \tag{16}$$

Upon differentiating the outputs once, it is seen that the partial relative degrees are all 1

$$\dot{y}_1 = f_1 C_{\mathrm{vl}} + f_2 C_{\mathrm{vg}} + f_3 C_{\mathrm{ve}}, \tag{17}$$

$$\dot{y}_2 = g_1 C_{\mathrm{vl}} + g_2 C_{\mathrm{vg}} + g_3 C_{\mathrm{ve}}, \tag{18}$$

$$\begin{aligned}\dot{y}_3 = -V\Bigg[vf_3 + C_{\mathrm{ve}}\Bigg(&\frac{\partial f_3}{\partial P}\frac{\partial P}{\partial \rho}(f_1 C_{\mathrm{vl}} + f_2 C_{\mathrm{vg}} + f_3 C_{\mathrm{ve}}) + \frac{\partial f_3}{\partial P}\frac{\partial P}{\partial u}(g_1 C_{\mathrm{vl}} + g_2 C_{\mathrm{vg}} + g_3 C_{\mathrm{ve}})\\ &+\frac{\partial f_3}{\partial \rho}(f_1 C_{\mathrm{vl}} + f_2 C_{\mathrm{vg}} + f_3 C_{\mathrm{ve}})\Bigg)\Bigg].\end{aligned} \tag{19}$$

Upon rearranging, the output derivatives can be expressed compactly as

$$\dot{y} = D + Ew,$$

where $\dot{y} = [\,\dot{y}_1 \quad \dot{y}_2 \quad \dot{y}_3\,]^{\mathrm{T}}$, $w = [\,C_{\mathrm{vl}} \quad C_{\mathrm{vg}} \quad v\,]^{\mathrm{T}}$ and

$$D = \begin{bmatrix} f_3 C_{\mathrm{ve}} \\ g_3 C_{\mathrm{ve}} \\ -VC_{\mathrm{ve}}^2\left[\frac{\partial f_3}{\partial P}\left(\frac{\partial P}{\partial \rho} f_3 + \frac{\partial P}{\partial u} g_3\right) + \frac{\partial f_3}{\partial \rho} f_3\right] \end{bmatrix},$$

$$E = \begin{bmatrix} f_1 & f_2 & 0 \\ g_1 & g_2 & 0 \\ -VC_{\mathrm{ve}}\left[\frac{\partial f_3}{\partial P}\left(\frac{\partial P}{\partial \rho} f_1 + \frac{\partial P}{\partial u} g_1\right) + \frac{\partial f_3}{\partial \rho} f_1\right] & -VC_{\mathrm{ve}}\left[\frac{\partial f_3}{\partial P}\left(\frac{\partial P}{\partial \rho} f_2 + \frac{\partial P}{\partial u} g_2\right) + \frac{\partial f_3}{\partial \rho} f_2\right] & -Vf_3 \end{bmatrix}.$$

Provided E is invertible in a region Ω of the state space, the feedback law

$$w = E^{-1}(\dot{y}_d - \Gamma(y - y_d) - D) \tag{20}$$

achieves exact linearization of the system, with tracking error dynamics given by

$$(\dot{y} - \dot{y}_d) + \Gamma(y - y_d) = 0.$$

If Γ is chosen as a diagonal positive-definite matrix, the resulting control law is called "decoupling control", since the dynamics of the output errors are decoupled. If the appropriate function definitions are substituted, the forms of the E and D matrices are as follows:

$$D = \begin{bmatrix} C_3' \sqrt{(P - P_{\mathrm{s}})\rho} C_{\mathrm{ve}} \\ C_3'' \sqrt{\frac{P - P_{\mathrm{s}}}{\rho}} \frac{P}{\rho} C_{\mathrm{ve}} \\ -\frac{VC_{\mathrm{ve}}^2 C_3'^2}{2}\left[\rho \frac{\partial P}{\partial \rho} + C \frac{P}{\rho} \frac{\partial P}{\partial u} + (P - P_{\mathrm{s}})\right] \end{bmatrix},$$

$$E = \begin{bmatrix} C_1' \sqrt{(P_1 - P)\rho_1} & f_2 & 0 \\ C_1' \sqrt{(P_1 - P)\rho_1}\left(\frac{h_1 - u}{\rho}\right) & f_2\left(\frac{h_{\mathrm{g}} - u}{\rho}\right) & 0 \\ -\frac{VC_{\mathrm{ve}} C_1' C_3' \sqrt{(P_1 - P)\rho_1}}{2\sqrt{(P - P_{\mathrm{s}})\rho}} \Gamma_1 & -\frac{VC_{\mathrm{ve}} f_2 C_3'}{2\sqrt{(P - P_{\mathrm{s}})\rho}} \Gamma_{\mathrm{g}} & -VC_3' \sqrt{(P - P_{\mathrm{s}})\rho} \end{bmatrix},$$

where

$$\Gamma_1 = \left[\rho \frac{\partial P}{\partial \rho} + (h_1 - u) \frac{\partial P}{\partial u} + (P - P_{\mathrm{s}})\right],$$

$$\Gamma_{\mathrm{g}} = \left[\rho \frac{\partial P}{\partial \rho} + (h_{\mathrm{g}} - u) \frac{\partial P}{\partial u} + (P - P_{\mathrm{s}})\right].$$

4.3. Input–output linearizability of augmented model

The ability to construct a feedback linearization controller hinges, first, on stable zero dynamics of the augmented system, and, second, on the invertibility of matrix E.

4.3.1. Invertibility of E

By inspection, it is readily seen that the first two rows of E are linearly independent provided $h_g \neq h_l$. This has a direct physical interpretation: if the two fluids have the same thermal properties (i.e., enthalpies), the ability to change the thermal properties of the mixture by changing the relative flows is lost. Gas and liquid enthalpies are different for the expected operating conditions. The third row is linearly independent from the first two if $P - P_s > 0$, which is also true for the mixer. Therefore E is invertible over the whole range of expected mixer operating conditions.

4.3.2. Zero dynamics of the augmented model

As it is known, the zero dynamics is preserved under input transformations [6]. This implies that one may examine the non-augmented model for zero dynamics and draw conclusions about the augmented model's zero dynamics. Suppose the exit flow is to be held constant at a value Y_{30}. The only way in which this can be achieved is by letting

$$C_{ve}(t) = -\frac{Y_{30}}{Vf_3(t)} \tag{21}$$

at all times. Differentiating the other two outputs and equating them to zero results in

$$C_{vl}(t) = \frac{1}{f_1(t)}\left(-f_2(t)C_{vg}(t) + \frac{Y_{30}}{V}\right), \tag{22}$$

$$C_{vg}(t) = -\frac{1}{g_2(t)}\left(g_1(t)C_{vl}(t) + \frac{g_3(t)Y_{30}}{Vf_3(t)}\right). \tag{23}$$

Upon substitution and rearrangement, it is seen that there exist three uniquely defined control inputs which hold the outputs constant. Since two of the outputs coincide with the states, it trivially follows that they are kept bounded, and therefore the system has stable zero dynamics (i.e., the system is minimum-phase). The control inputs are given by Eq. (21) and, dropping the time notation,

$$C_{vg} = \frac{\frac{Y_{30}f_1}{V}\left(\frac{g_3}{f_3} - \frac{g_1}{f_1}\right)}{f_1g_2 - g_1f_2}, \tag{24}$$

$$C_{vl} = \frac{Y_{30}}{Vf_1}\left[1 - \frac{f_1f_2}{f_1g_2 - g_1f_2}\left(\frac{g_3}{f_3} - \frac{g_1}{f_1}\right)\right]. \tag{25}$$

Note that the above formulas can be used to find the valve positions at which the system has a prescribed outflow and a pair of thermodynamic properties.

5. Small-signal model

The dynamic model of the mixer can be written in the form

$$\dot{z}_1 = F_1(z_1, z_2, C_{\mathrm{vg}}, C_{\mathrm{vl}}, C_{\mathrm{ve}}), \tag{26}$$

$$\dot{z}_2 = F_2(z_1, z_2, C_{\mathrm{vg}}, C_{\mathrm{vl}}, C_{\mathrm{ve}}), \tag{27}$$

where $F_1(\cdot)$ and $F_2(\cdot)$ are nonlinear functions of the state and valve coefficients. For the remainder of this article, we will denote constant or equilibrium values of any variable by an upper bar $\bar{(\cdot)}$. Given constant values of valve flow coefficients $\overline{C}_{\mathrm{v}} = [\overline{C}_{\mathrm{vg}}\ \overline{C}_{\mathrm{vl}}\ \overline{C}_{\mathrm{ve}}]^{\mathrm{T}}$ (superscript T denotes transposition) and constant fluid properties, the state of the model

$$z(t) = \begin{bmatrix} z_1: & \text{Density} \\ z_2: & \text{Internal Energy} \end{bmatrix}$$

reaches a constant equilibrium point $\bar{z}$. Next, consider perturbing such an equilibrium by small signals $x(t)$ and $\Delta C_{\mathrm{v}}(t)$ so that

$$z(t) = \bar{z} + x(t) \quad \text{and} \quad C_{\mathrm{v}}(t) = \overline{C}_{\mathrm{v}} + \Delta C_{\mathrm{v}}(t),$$

where $\Delta C_{\mathrm{v}}(t)$ denotes a small, valve-coefficient correction/regulation signal. Then, a standard linearization of Eqs. (26), (27) results in the small-signal model

$$\dot{x} = Ax + B\Delta C_{\mathrm{v}}, \tag{28}$$

where $x(t)$ is the small perturbation state vector, $\Delta C_{\mathrm{v}}(t)$ is the small perturbation control signal, and the two-by-two matrix A and two-by-three matrix B are given by

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} \\ \frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} \end{bmatrix}; \quad B = \begin{bmatrix} \frac{\partial F_1}{\partial C_{\mathrm{vg}}} & \frac{\partial F_1}{\partial C_{\mathrm{vl}}} & \frac{\partial F_1}{\partial C_{\mathrm{ve}}} \\ \frac{\partial F_2}{\partial C_{\mathrm{vg}}} & \frac{\partial F_2}{\partial C_{\mathrm{vl}}} & \frac{\partial F_2}{\partial C_{\mathrm{ve}}} \end{bmatrix},$$

where the partial derivatives are evaluated at the equilibrium state $\bar{z}$ and constant valve flow coefficient vector $\overline{C}_{\mathrm{v}}$. Output equations of the form

$$y = Cx + D\Delta C_{\mathrm{v}},$$

where matrices C and D are appropriately dimensioned are easily appended to the model (28) to account for the measurement of certain variables such as temperature, pressure, or flow. An output of interest is the exit flow w_{e}. Using Eq. (3) and linearizing around a chosen equilibrium point, we obtain the linear approximation

$$w_{\mathrm{elin}} = \left[\frac{\partial \alpha_{\mathrm{e}}}{\partial z_1} \quad \frac{\partial \alpha_{\mathrm{e}}}{\partial z_2}\right]_{\mathrm{eq}} x + [\alpha_{\mathrm{e}}]_{\mathrm{eq}} \Delta C_{\mathrm{ve}} = C_{\mathrm{vflow}} x + D_{\mathrm{flow}} \Delta C_{\mathrm{ve}}.$$

Other outputs of interest are the mixer pressure P_{v} and the exit temperature T_{e}. Taking these to be functions of the mixer internal states, that is,

$$P_{\mathrm{v}} = f_P(z_1(t), z_2(t)) \quad \text{and} \quad T_{\mathrm{e}} = f_T(z_1(t), z_2(t))$$

Table 1
Initial Equilibrium Data. z_1 in kg/m^3; z_2 in kJ/kg

	$\overline{P}$ (MPa)	$\overline{T}$ (K)	$\bar{\rho}$ (kg/m^3)	$\bar{h}$ (kJ/kg)	C_v	$\bar{w}$ (kg/s)	$\bar{z}_1$	$\bar{z}_2$
GH2	94	305	46.90	5171	2.01	1.552	–	–
LH2	59	66	81.11	1019	20.57	15.45	–	–
Mixer	47	101	62.45	1398	–	–	62.45	645.8
Outflow	38	105	55.24	1398	29.82	17	–	–

then, the linearization around the chosen equilibrium gives

$$P_{\text{vlin}} = \left[\frac{\partial P_v}{\partial z_1} \quad \frac{\partial P_v}{\partial z_2}\right]_{\text{eq}} x = C_{pv}x$$

and

$$T_{\text{elin}} = \left[\frac{\partial T_e}{\partial z_1} \quad \frac{\partial T_e}{\partial z_2}\right]_{\text{eq}} x = C_{te}x.$$

The indicated partial derivative terms may be calculated numerically using accessory routines.

5.1. Numerical example

Table 1 lists the data for a typical equilibrium point EQ_i corresponding to a given set of outflow requirements. The working substance in this example is Hydrogen. The matrices A and B are evaluated at the equilibrium EQ_i listed in Table 1 to be

$$A = \begin{bmatrix} -43.806 & -1.2337 \\ -493.48 & -19.065 \end{bmatrix}; \quad B = \begin{bmatrix} 10.595 & 10.661 & -8.0509 \\ 63.366 & 772.547 & -97.023 \end{bmatrix}.$$

For an output $y = w_e$, the linear approximation is

$$w_{\text{elin}} = Cx + DC_{\text{ve}} = [\,-26.831 \quad -0.734\,]x - [8.081]\Delta C_{\text{ve}}.$$

The eigenvalues of A lie in the left half of the complex plane, establishing the local stability of the model by Lyapunov's linearization method [17]. Although not shown here, the computation can be performed at any point in the range of operation, showing stability in a larger region. Local controllability [17] can be likewise evaluated from the linearization at various points in the operating region. In this example, the controllability matrix $[B \mid AB]$ has rank two.

6. Model validation

The usefulness of the model must be demonstrated with real data. Visual and quantitative assessments of the model's ability to capture the essential dynamics of mixer operation are provided in this section. Data available for model validation consisted of a preliminary operation of the mixer with Nitrogen as working substance. The thermodynamic lookup routines discussed earlier were set to work with Nitrogen, without modifications to the model equations.

6.1. Calibration of valve coefficients

Part of the experimental data corresponds to nearly steady conditions. A measurement of the flows and of some thermodynamic properties can then be used to estimate the corresponding values of the model valve coefficients. The coefficients corresponding to other percent openings are found by direct proportionality.

6.2. Boundary conditions and valve position histories

The thermodynamic properties at the inlet of the liquid and gas valves change with time and were measured through the duration of the experiment. In particular, pressure and temperature were recorded, data from which the enthalpy and density can be easily found by direct lookup of appropriate thermodynamic tables. The exit fluid was dumped into the atmosphere, thus establishing a fixed boundary pressure at the outlet of the exit valve. Knowledge of the above properties, valve position history and initial conditions of the mixer states is sufficient to obtain a numerical solution to the differential equations. The initial conditions are obtained from the measured mixer internal temperature and pressure at the beginning of the experiment. These mixer variables were measured at all subsequent times during the experiment and provide the basis for

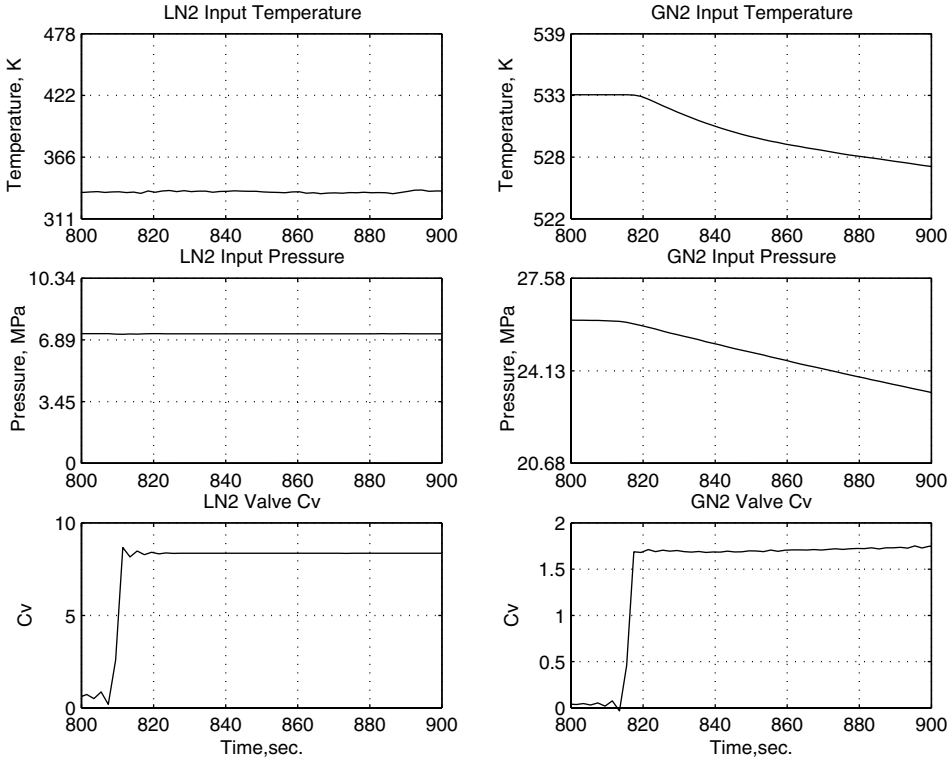

Fig. 3. Inlet thermodynamic properties and valve position history.

comparison and evaluation of the model. It is to be noted that during the actual experiment, the exit valve was kept at a fixed position. Fig. 3 shows the measured temperatures and pressures of the liquid and gas nitrogen as recorded, as well as the valve positions.

6.3. Simulation

The boundary variables shown in Fig. 3 were fed into a Simulink model of the mixer equations. Figs. 4 and 5 show the simulated and measured temperature and pressure inside the mixer. Despite of some spurious effects due to the interaction of the integration method with noisy input data, it can be seen that the agreement is very good. It should be noted that the pressure curve has been slightly shifted to compensate for an initial offset error found in the original data.

6.4. Immediate improvements to the model

The mathematical model developed may be improved and extended by considering the static and dynamic characteristics of valve actuation. In the above development, it is assumed that the valve opening coefficients can be directly and instantaneously commanded to any desired value. This assumption did not interfere, however, with the model validation, since actual valve coefficient responses were available and were used instead of valve commands. In the practical

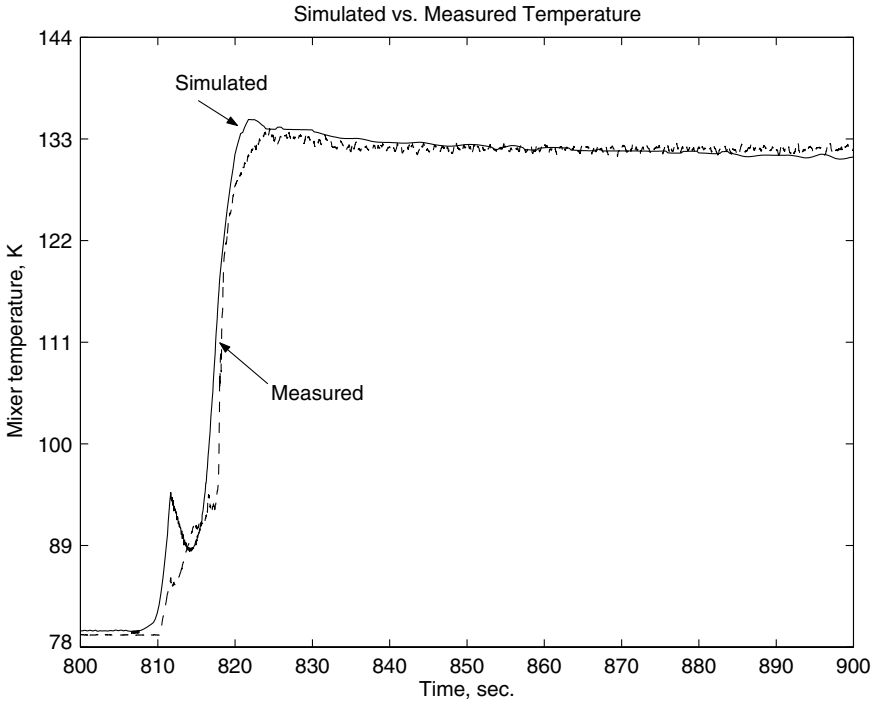

Fig. 4. Temperature validation.

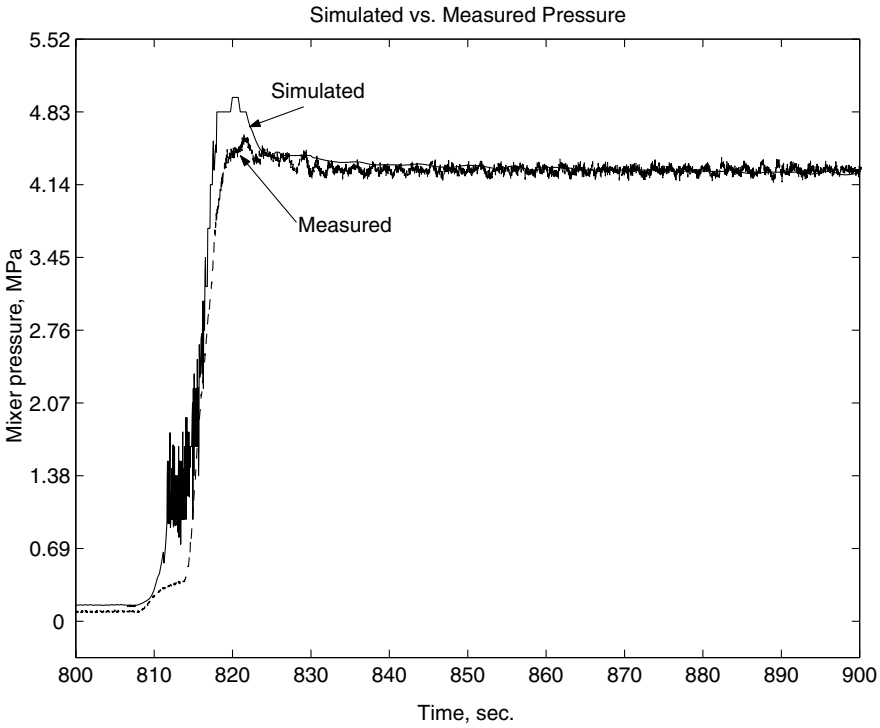

Fig. 5. Pressure validation.

situation, the control system has authority over the commanded openings only. A simple model of valve response is given by the first-order system

$$\dot{C}_{\mathrm{v}} = -\frac{1}{\tau} C_{\mathrm{v}} + \beta\zeta, \tag{29}$$

where ζ is the commanded valve opening, τ is a time constant and β is a proportionality constant. An equation of the above form must be written for each of the three valves in the system and the model must be augmented with these dynamics, thus becoming a fifth-order model. Note that direct proportionality has been assumed to exist between ζ and C_{v} under steady conditions ($\dot{C}_{\mathrm{v}} = 0$). If nonlinearity is observed in an experimental characterization of the valve, it can be included in the form

$$\dot{C}_{\mathrm{v}} = -\frac{1}{\tau} C_{\mathrm{v}} + g(\zeta), \tag{30}$$

where g is a function to be determined from static calibration, as done in [13]. Further improvements to the model include adding a disturbance term—mainly corresponding to unmodelled heat transfer—to be used in robust controller design. A Matlab-based graphical user interface (GUI) has been developed that provides flexibility in using the model in simulation studies [9,12].

7. Conclusions

A mathematical model for the mixer system is presented. The model consists of a system of two nonlinear equations having density and internal energy as states. The independent variables are the valve positions represented by flow coefficients, and the controlled outputs are mixer pressure, exit temperature and exit mass flow. Model reliability is ensured by the use of function calls to real thermodynamic data. It is shown that, in the expected range of operation, the mixer has a single equilibrium point for each set of fixed valve coefficients and that it is feedback linearizable and locally stable. The reachability of operating conditions is shown by the possibility of determining a unique set of valve coefficients that achieve a desired set of steady outputs. The model is validated against experimental data with excellent agreement.

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